

ON THE DEGREE OF CONVERGENCE OF THE GRAM-CHARLIER SERIES*

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The degree of convergence of Fourier series and of polynomial approximations has been extensively studied by Jackson,[†] and similar investigations have been made for series of Legendre polynomials,[‡] for Sturm-Liouville series,[§] for Laplace's series,^{||} for Birkhoff's series,[¶] and for other related problems.** In all these cases the interval over which the convergence applies is finite. It is therefore of interest to inquire what are the corresponding facts in a case where the interval is infinite. Such a case is afforded by the Gram-Charlier series, which, in addition to having an infinite interval, is of interest and importance as a mathematical statement of the law of errors^{††} and because of its application to the representation of frequency functions.

In the following pages we propose to examine the degree of convergence of the Gram-Charlier series and associated expansions, principally from the point of view of developments in characteristic solutions of homogeneous linear differential systems, a procedure which correlates the work closely with the previously cited papers on degree of convergence. The results obtained indicate that the Gram-Charlier series converges in general more slowly than the Fourier series for a similar function in a finite interval. Roughly speaking we may summarize the situation as follows: when the remainder after n terms of a Fourier series is $O(1/n^k)$ the remainder after n terms of the Gram-Charlier series is $O(1/n^{k/2})$.

1. If we denote the normal probability function by $\phi_0(x)$ and its derivative of order n by $\phi_n(x)$, then

$$(1) \quad \phi_0(x) = (2\pi)^{-1/2}e^{-x^2/2}, \quad \phi_n(x) = H_n(x)\phi_0(x),$$

* Presented to the Society, March 30 and June 21, 1929; received by the editors March 7, 1929.

† Jackson, these Transactions, vol. 13 (1912), pp. 491-515.

‡ Jackson, these Transactions, vol. 13 (1912), pp. 305-318.

§ Jackson, these Transactions, vol. 15 (1914), pp. 439-466.

|| Gronwall, these Transactions, vol. 15 (1914), pp. 1-30.

¶ Milne, these Transactions, vol. 19 (1918), pp. 143-156.

** Jackson, these Transactions, vol. 14 (1913), pp. 343-364, and vol. 22 (1921), pp. 158-166. See also Bulletin of the American Mathematical Society, vol. 27 (1921), pp. 415-431.

†† Charlier, Arkiv för Matematik, Astronomi, och Fysik, vol. 2 (1905).

where $H_n(x)$ is a polynomial of degree n , called an Hermite polynomial.* The functions $\phi_m(x)$ and $H_n(x)$ are biorthogonal in the infinite interval, so that

$$(2) \quad \int_{-\infty}^{\infty} \phi_m(x) H_n(x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ n! & \text{if } m = n. \end{cases}$$

Hence we have the formal expansion of an arbitrary function $\phi(x)$

$$(3) \quad \phi(x) = a_0 \phi_0(x) + a_1 \phi_1(x) + a_2 \phi_2(x) + \dots$$

in which the coefficients are determined by the formulas

$$(4) \quad a_n = (1/n!) \int_{-\infty}^{\infty} \phi(x) H_n(x) dx.$$

The series (3) is the Gram-Charlier series. The convergence of this and related series has been investigated by Myller-Lebedeff,† Watson,‡ Cramér,§ Szegő,|| Rotach,¶ Hille,** Stone,†† and others.

The functions $\phi_n(x)$ and $H_n(x)$ respectively satisfy the adjoint differential equations

$$(5) \quad u'' + xu' + (\lambda + 1)u = 0,$$

and

$$(6) \quad v'' - xv' + \lambda v = 0,$$

for integral values of λ , $\lambda = n$. If v is any solution of (6) the function

$$u = e^{-x^2/2}v$$

is a solution (5) and conversely. The transformations

$$u = e^{-x^2/4}w, \quad v = e^{x^2/4}w$$

carry (5) and (6) respectively into the self-adjoint equation‡‡

* Hermite, *Comptes Rendues*, vol. 58, p. 93.

† Myller-Lebedeff, *Mathematische Annalen*, vol. 64 (1907), p. 388.

‡ Watson, *Proceedings of the London Mathematical Society*, (2), vol. 8 (1910), pp. 393–421.

§ Cramér, *Den Sjette Skandinaviske Matematikerkongres i København, Kongresberetningen*, Copenhagen, 1926, pp. 399–425.

|| Szegő, *Mathematische Zeitschrift*, vol. 25 (1926), pp. 112–115.

¶ Rotach, *Promotionsarbeit*, Zurich, Ceuf, 1925, 33 pp.

** Hille, *Annals of Mathematics*, (2), vol. 27 (1926), pp. 427–464.

†† Stone, *Annals of Mathematics*, (2), vol. 29 (1928), pp. 1–11.

‡‡ Equation (7) is sometimes called Weber's equation, and is satisfied by the harmonic functions associated with the parabolic cylinder. See Weber, *Mathematische Annalen*, vol. 1, p. 1; Watson, loc. cit.; Whittaker, *Proceedings of the London Mathematical Society*, vol. 35 (1902), p. 417.

$$(7) \quad w'' + (\lambda + 1/2 - x^2/4)w = 0.$$

Corresponding to the three equations (5), (6), and (7) there are three distinct expansions in terms of characteristic functions, each one of which can be transformed into any other by means of the above relationships connecting u , v , and w .

2. Instead of dealing directly with the series (3) it is convenient to use as the basis of the following investigation the set of normalized orthogonal solutions of (7),

$$W_0(x), W_1(x), W_2(x), \dots,$$

defined by the equation

$$(8) \quad W_n(x) = (2\pi)^{-1/4} (n!)^{-1/2} e^{-x^2/4} H_n(x).$$

From the well known properties of Hermite polynomials we can easily write down a number of important formulas as follows:

$$(9) \quad (n+1)^{1/2} W_{n+1}(x) + x W_n(x) + n^{1/2} W_{n-1}(x) = 0;$$

$$(10) \quad W'_n(x) = -(x/2) W_n(x) - n^{1/2} W_{n-1}(x);$$

$$(11) \quad W'_n(x) = (1/2) [(n+1)^{1/2} W_{n+1}(x) - n^{1/2} W_{n-1}(x)];$$

$$(12) \quad W''_n(x) + (n+1/2 - x^2/4) W_n(x) = 0;$$

$$(13) \quad \sum_{m=0}^{n-1} W_m(x) W_m(s) = n^{1/2} \{ W_{n-1}(x) W_n(s) - W_n(x) W_{n-1}(s) \} (x-s)^{-1};$$

$$(14) \quad \int_{-\infty}^{\infty} W_m(x) W_n(x) dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

From equation (11),

$$W_{n+1}(x) = (n+1)^{-1/2} \{ 2W'_n(x) + n^{1/2} W_{n-1}(x) \}.$$

If we form a similar equation for $W_{n-1}(x)$ and substitute its value into the above expression for $W_{n+1}(x)$, then substitute again for $W_{n-3}(x)$, and continue the process the following two formulas are obtained (in which we have changed subscripts from $n+1$ to n), the first for the case in which n is even and the second for the case in which n is odd:

$$(15) \quad W_n(x) = 2[n^{-1/2} W'_{n-1}(x) + \{(n-1)/(n(n-2))\}^{1/2} W'_{n-3}(x) + \dots \\ + \{(n-1) \dots 7 \cdot 5 \cdot 3 / (n \dots 6 \cdot 4 \cdot 2)\}^{1/2} W'_1(x) \\ + \{(n-1) \dots 7 \cdot 5 \cdot 3 \cdot 1 / (n \dots 6 \cdot 4 \cdot 2)\}^{1/2} e^{-x^2/4} (2\pi)^{-1/4};$$

$$(16) \quad W_n(x) = 2[n^{-1/2} W'_{n-1}(x) + \{(n-1)/(n(n-2))\}^{1/2} W'_{n-3}(x) + \dots \\ + \{(n-1) \dots 6 \cdot 4 \cdot 2 / (n \dots 5 \cdot 3 \cdot 1)\}^{1/2} W'_0(x)].$$

Since the W 's with odd subscripts vanish at 0 and ∞ we find from (15)

$$(17) \quad \int_0^\infty W_n(x)dx = (\pi/2)^{1/4} \{1 \cdot 3 \cdot 5 \cdots (n-1)/(2 \cdot 4 \cdot 6 \cdots n)\}^{1/2},$$

when n is even. On the other hand when n is odd (16) may be written

$$(18) \quad \begin{aligned} W_n(x) = 2 \{2 \cdot 4 \cdot 6 \cdots (n-1)/(1 \cdot 3 \cdot 5 \cdots n)\}^{1/2} [W_0'(x) + (1/2)^{1/2} W_2'(x) \\ + (1 \cdot 3/(2 \cdot 4))^{1/2} W_4'(x) + \cdots \\ + \{1 \cdot 3 \cdot 5 \cdots (n-2)/(2 \cdot 4 \cdot 6 \cdots (n-1))\}^{1/2} W_{n-1}'(x)]. \end{aligned}$$

When m is even, $W_m(x)$ vanishes at ∞ , but at $x=0$ we have

$$(19) \quad W_m(0) = (-1)^{m/2} (2\pi)^{-1/4} (1 \cdot 3 \cdot 5 \cdots (m-1)/(2 \cdot 4 \cdot 6 \cdots m))^{1/2},$$

so that

$$(20) \quad \begin{aligned} \int_0^\infty W_n(x)dx = -2(2\pi)^{-1/4} [2 \cdot 4 \cdot 6 \cdots (n-1)/(1 \cdot 3 \cdot 5 \cdots n)]^{1/2} \\ \times [1 - 1/2 + 1 \cdot 3/(2 \cdot 4) - 1 \cdot 3 \cdot 5/(2 \cdot 4 \cdot 6) + \cdots \\ \pm 1 \cdot 3 \cdot 5 \cdots (n-2)/(2 \cdot 4 \cdot 6 \cdots (n-1))] \end{aligned}$$

when n is odd. The series in brackets is the series, up to terms of degree $n-1$, for the expansion of $(1+x^2)^{-1/2}$ at the value $x=1$, and therefore the sum of the terms in brackets in (20) differs from $2^{-1/2}$ by a quantity less than the last term. Stirling's formula for $n!$ enables us to estimate the magnitude of this last term, which we find to be $(2/(\pi n))^{1/2}(1+O(n^{-1}))$. Therefore (20) may be rewritten

$$(21) \quad \int_0^\infty W_n(x)dx = - (2/\pi)^{1/4} [2 \cdot 4 \cdot 6 \cdots (n-1)/(1 \cdot 3 \cdot 5 \cdots n)]^{1/2} (1 + \epsilon),$$

when n is odd.

Finally if (17) and (21) are evaluated by means of Stirling's formula we have

$$(22) \quad \int_0^\infty W_n(x)dx = \pm n^{-1/4} (1 + \epsilon_n), \quad + \text{ if } n \text{ is even, } - \text{ if } n \text{ is odd.}$$

3. To make clear the subsequent discussion it is needful to describe in some detail the character of the function $W_n(x)$. First of all $W_n(x)$ is even when n is even and odd when n is odd, so that we may limit the discussion to the case where x is positive. By well known theorems of oscillation we see from the differential equation (7) that in the interval $0 \leq x < h$ (where for brevity we let $h = (4n+2)^{1/2}$) the function $W_n(x)$ oscillates with increasing amplitudes, increasing intervals between the roots, and with decreasing slopes

at the roots. In the interval $h < x < \infty$, $W_n(x)$ does not oscillate, but approaches zero to an infinite order as x becomes infinite. The roots of $W_n(x)$ are identical with the roots of $H_n(x)$, which are known to be all real, n in number, symmetrically placed with respect to the origin, and all in the interval $-h < x < h$. The largest maximum of $W_n(x)$ occurs in the interval $r < x < h$, where r denotes the largest root. The approximate location of this largest root may be estimated as follows. In (7) we make the substitution

$$x = h - (2/h)^{1/3}t,$$

which reduces it to

$$(23) \quad d^2w/dt^2 + t[1 - (2/h)^{4/3}t/4]w = 0.$$

We now form two comparison equations

$$d^2u/dt^2 + tu = 0,$$

and

$$d^2v/dt^2 + (t/2)v = 0.$$

We solve (23) with those conditions at $t=0$ which furnish the solution $W_n(x)$ and solve the two comparison equations with the same initial values. We denote the first positive roots of w , u , and v by t_2 , t_1 , and t_3 respectively. Then if n is large enough that

$$1 - (2/h)^{4/3}t_3/4 > 1/2$$

we see by comparing the differential equations that

$$t_1 < t_2 < t_3.$$

Now t_1 and t_3 are not entirely independent of n because of the initial conditions. But at $t=0$, u and du/dt have opposite signs, from which we may conclude that t_1 has a lower bound c_1 , not zero, and independent of n . By numerical calculation we find from the equation $d^2u/dt^2 + tu = 0$ that it is possible to take $c_1 = 7^{1/3} = 1.91$. Also t_3 has a finite upper bound c_3 , independent of n . Consequently

$$(24) \quad h - (2/h)^{1/3}c_3 < r < h - (2/h)^{1/3}c_1.$$

To the function $W_n(x)$ may be applied certain inequalities based on Hille's investigations* in the case of Hermitian polynomials. Following his methods we are able to show that

$$(25) \quad |W_n(x)| < K(h^2 - x^2)^{-1/4}, \quad -r \leq x \leq r,$$

where K is a constant independent of n . We have also

$$(26) \quad |W_n(x)| < K_1 n^{-1/12}$$

* Hille, loc. cit., pp. 433-436.

for all values of x . A further inequality applicable to the interval $h < x < \infty$ may be established by the following considerations: When n is even, the function $W_n(x)$ is positive and decreasing and its curve is concave upward for all values of x greater than h . If we denote by $w_0, w_1, w_2, \dots, w'_0, w'_1, w'_2, \dots$ the values of $W_n(x)$ and $W'_n(x)$ at the points $h, h+1, h+2, \dots$, the following inequalities are apparent:

$$(27) \quad w_{k-1} > -w'_k.$$

From the equation (7) we see that, in the interval from $h+k$ to $h+k+1$,

$$w'' > \frac{1}{4}[(h+k)^2 - h^2]w_{k+1}.$$

After integrating from $h+k$ to $h+k+1$ and dropping some terms to strengthen the inequality we have

$$(28) \quad -w'_k > \frac{1}{2}khw_{k+1}.$$

From (27) and (28) it follows that

$$w_{k+1} < 2w_{k-1}/(kh)$$

and from this we get by successive substitutions

$$w_2 < 2w_0/h, \quad w_4 < 4w_0/(3h^2), \quad w_6 < 8w_0/(3 \cdot 5h^3), \dots$$

Therefore we may conclude that for $x \geq h+2k$

$$(29) \quad |W_n(x)| < K_2 n^{-k/2}.$$

The case in which n is odd is similarly treated.

4. In addition to estimates of the magnitude of $W_n(x)$ we shall require estimates of the magnitude of the integral of $W_n(x)$. For this purpose let r_1, r_2, r_3, \dots, r denote the positive roots of $W_n(x)$. We then have the inequality

$$(30) \quad \left| \int_{r_{i-1}}^{r_i} W_n(x) dx \right| < \left| \int_{r_i}^{r_{i+1}} W_n(x) dx \right|.$$

Let

$$m_i = \pi/(r_{i+1} - r_i).$$

Then from (12) we have

$$(h^2 - r_{i+1}^2)^{1/2} < 2m_i < (h^2 - r_i^2)^{1/2},$$

as may be seen by comparing the intervals between successive roots of those solutions of the differential equations

$$d^2u/dx^2 + \frac{1}{4}(h^2 - r_{i+1}^2)u = 0,$$

$$d^2u/dx^2 + \frac{1}{4}(h^2 - x^2)u = 0,$$

$$d^2u/dx^2 + \frac{1}{4}(h^2 - r_i^2)u = 0,$$

which vanish at $x = r_i$.

Hence there is a value x_i of x between r_i and r_{i+1} such that

$$(h^2 - x_i^2)^{1/2} = 2m_i.$$

The equation (12) may now be written in the form

$$W_n''(x) + m_i^2 W_n(x) = \frac{1}{4}(x^2 - x_i^2)W_n(x),$$

from which we obtain the integral equation

$$(31) \quad \begin{aligned} W_n(x) = & (1/m_i)W_n'(r_{i+1}) \sin m_i(x - r_{i+1}) \\ & + \frac{1}{4m_i} \int_{r_{i+1}}^x \{\sin m_i(x - s)\} (s^2 - x_i^2) W_n(s) ds. \end{aligned}$$

The integral of this from r_i to r_{i+1} is

$$\begin{aligned} \int_{r_i}^{r_{i+1}} W_n(x) dx = & - (1/m_i)^2 2W_n'(r_{i+1}) \\ & + \frac{1}{4m_i} \int_{r_i}^{r_{i+1}} dx \int_{r_{i+1}}^x \{\sin m_i(x - s)\} (s^2 - x_i^2) W_n(s) ds. \end{aligned}$$

Now

$$|s^2 - x_i^2| < r_{i+1}^2 - r_i^2 < 2\pi r_{i+1}/m_i < 4\pi r_{i+1}(h^2 - r_{i+1}^2)^{-1/2},$$

and from (25)

$$|W_n(s)| < K(h^2 - r_{i+1}^2)^{-1/4}$$

so that using these inequalities together with the inequalities for m_i we have

$$(32) \quad \left| \frac{1}{4m_i} \int_{r_i}^{r_{i+1}} dx \int_{r_{i+1}}^x \{\sin m_i(x - s)\} (s^2 - x_i^2) W_n(s) ds \right| < C_2 r_{i+1} (h^2 - r_{i+1}^2)^{-9/4},$$

where C_2 is a constant independent of n .

It will be found upon reference to Hille's arguments from which (25) was derived that we also have the inequality

$$(33) \quad |W_n'(x)| < K'(h^2 - x^2)^{1/4},$$

where K' is independent of n , and x is between $-r$ and r . Then using (33) in connection with (32) we are led to the inequality

$$(34) \quad \left| \int_{r_i}^{r_{i+1}} W_n(x) dx \right| < 8K'(h^2 - r_{i+1}^2)^{-3/4} + C_2 r_{i+1} (h^2 - r_{i+1}^2)^{-9/4}.$$

From the fact that the successive integrals satisfy the inequalities (30) and alternate in sign we see at once that

$$(35) \quad \left| \int_0^x W_n(x) dx \right| < 8K'(h^2 - r_i^2)^{-3/4} + C_2 r_i (h^2 - r_i^2)^{-9/4}, \quad x \leq r_i.$$

In particular if x is in the interval $-N \leq x \leq N$, where $N = (3n+2)^{1/2}$,

$$(36) \quad \left| \int_0^x W_n(x) dx \right| < 8K'n^{-3/4} + O(n^{-7/4}).$$

Since the extrema of the integral occur only at the roots of $W_n(x)$ its largest maximum absolute value must be found either at one of the roots or else at $x = \infty$. If the former is true we find from (35) together with the fact that the largest root is limited by (24) that

$$(37) \quad \left| \int_0^x W_n(x) dx \right| < 2K'n^{-1/4}[1 + \epsilon],$$

for all values of x . On the other hand if the largest maximum occurs at $x = \infty$ its value is given by (22).

5. We are now in a position to investigate questions concerning degree of convergence. First of all it will be interesting and instructive to consider a series formed for a finite interval and to watch what happens as the ends of the interval recede to $-\infty$ and ∞ . Together with equation (7) let us consider a pair of Sturmian boundary conditions at the ends of a finite interval $(-c, c)$, and let $w_n(x)$ denote the normalized orthogonal characteristic solution of this system corresponding to the $(n+1)$ th characteristic number λ_n . The formal expansion of an arbitrary function $f(x)$ will be

$$(38) \quad f(x) = c_0 w_0(x) + c_1 w_1(x) + c_2 w_2(x) + c_3 w_3(x) + \dots$$

This is a Sturm-Liouville series, so that the results obtained by Jackson regarding the degree of convergence of these series may be applied directly. We are told for example that if $f(x)$ has a continuous k th derivative of bounded variation in the interval $(-c, c)$, and if $f(x)$ and its first $k-1$ derivatives vanish at $-c$ and c , then the first n terms of the series (38) represents $f(x)$ with an error which is $O(\lambda_n^{-k/2})$.

When c is allowed to become infinite the characteristic number λ_{n+1} approaches n as a limit and each term of the series (38) approaches* the corresponding term of the series

* Cf. Weyl, Göttinger Nachrichten, 1910, p. 442; also Milne, these Transactions, vol. 30 (1928), pp. 797-802.

$$(39) \quad f(x) = C_0 W_0(x) + C_1 W_1(x) + C_2 W_2(x) + C_3 W_3(x) + \cdots$$

where

$$(40) \quad C_n = \int_{-\infty}^{\infty} f(x) W_n(x) dx.$$

We are thus led to conjecture that if $f(x)$ has a k th derivative of bounded variation in the infinite interval, and if $f(x)$ and its derivatives vanish at $\pm \infty$, then the first n terms of (39) will represent $f(x)$ with an error which is $O(n^{-k/2})$. The essential difference between this result and that for the finite interval lies in the distribution of the characteristic numbers in the two cases. For the finite interval the n th characteristic number is of the order of magnitude of $(\pi n/c)^2$, while for the infinite interval $\lambda_{n+1} = n$. Thus in the former case the remainder term is $O(n^{-k})$ while in the latter we conjecture that it is $O(n^{-k/2})$.

Subsequent theorems show that this conjecture is substantially correct.

6. Instead of attempting to carry out rigorously the line of thought suggested in the preceding paragraph we shall deal directly with the series (39). Letting $S_n(x)$ denote the sum of the first n terms of this series we may state the first theorem as follows:

THEOREM I. *If $f(x)$ has a continuous k th derivative of bounded variation, and if $xf^{(k)}(x)$, $x^2 f^{(k-1)}(x)$, \dots , $x^{k+1}f(x)$ also are of bounded variation in the infinite interval, then*

$$S_n(x) = f(x) + O(n^{-k/2}), \quad -N \leq x \leq N,$$

and

$$S_n(x) = f(x) + O(n^{-(3k-1)/6}), \quad -\infty < x < \infty.$$

These relations hold uniformly with respect to x in the specified intervals.

From (8) and (40) we may set

$$C_n = (2\pi)^{-1/4} (n!)^{-1/2} \int_{-\infty}^{\infty} e^{-x^2/4} f(x) H_n(x) dx,$$

whence integrating by parts k times with the aid of the relation

$$H_n(x) dx = -(n+1)^{-1} dH_{n+1}(x),$$

and observing that the integrated terms vanish at the limits, we obtain

$$C_n = [(n+1)(n+2) \cdots (n+k)]^{-1/2} \int_{-\infty}^{\infty} G_k(x) W_{n+k}(x) dx,$$

in which

$$G_k(x) = e^{x^2/4} \frac{d^k}{dx^k} [e^{-x^2/4} f(x)].$$

From the hypotheses it is clear first of all that $xG_k(x)$ is of bounded variation in the infinite interval, and since $f^{(k)}(x)$ is also of bounded variation it is easy to show that $G_k(x)$ is of bounded variation in the infinite interval. Hence we may set

$$G_k(x) = g_1(x) - g_2(x)$$

where $g_1(x)$ and $g_2(x)$ are positive or zero, continuous, monotone increasing, and bounded. By the second law of the mean

$$\int_{-N}^N G_k(x) W_{n+k}(x) dx = g_1(N) \int_{\xi_1}^N W_{n+k}(x) dx - g_2(N) \int_{\xi}^N W_{n+k}(x) dx$$

where ξ_1 and ξ_2 are in the interval $(-N, N)$. Therefore by (36)

$$\int_{-N}^N G_k(x) W_{n+k}(x) dx = O(n^{-3/4}).$$

Next we may write

$$\int_N^\infty G_k(x) W_{n+k}(x) dx = \int_N^\infty [xG_k(x)] [W_{n+k}(x)/x] dx.$$

Since $xG_k(x)$ is of bounded variation, and since

$$\begin{aligned} \int_{\xi}^\infty [W_{n+k}(x)/x] dx &= - (1/\xi) \int_0^{\xi} W_{n+k}(x) dx \\ &\quad + \int_{\xi}^\infty x^{-2} dx \int_0^x W_{n+k}(x) dx = O(n^{-3/4}) \end{aligned}$$

if $\xi > N$, we may deduce in the same manner as above that

$$\int_N^\infty G_k(x) W_{n+k}(x) dx = O(n^{-3/4}).$$

The same is true of the integral from $-\infty$ to $-N$, so that finally

$$\int_{-\infty}^\infty G_k(x) W_{n+k}(x) dx = O(n^{-3/4})$$

and

$$C_n = O(n^{-k/2-3/4}).$$

Therefore in the interval $(-N, N)$ the inequality (25) gives

$$C_n W_n(x) = O(n^{-k/2-1}),$$

and hence in the same interval

$$\sum_{m=n}^{\infty} C_m W_m(x) = O(n^{-k/2}).$$

In view of the fact that under the given hypotheses the series is known to converge to $f(x)$ the first conclusion stated by the theorem follows. To establish the second we have only to use the inequality (26) in place of (25).

In the statement and proof of the foregoing theorem continuity of the k th derivative has been assumed. An examination of the methods of proof will reveal that this restriction can be to some extent dispensed with. For example the results can still be shown to hold true even if $f^{(k)}(x)$ has a finite number of finite discontinuities.

It may also be observed that the other conditions of the hypotheses of the theorem can be variously stated. Any set of conditions that will insure the convergence of the series to $f(x)$ and at the same time make the functions $G_k(x)$ and $xG_k(x)$ of limited variation in the infinite interval will do.

From Theorem I we may derive directly a corresponding theorem applicable to the Gram-Charlier series (3).

THEOREM II. *If $\phi(x)$ has a continuous k th derivative of bounded variation, and if*

$$xe^{x^2/4}\phi^{(k)}(x), \quad x^2e^{x^2/4}\phi^{(k-1)}(x), \quad \dots, \quad x^{k+1}e^{x^2/4}\phi(x)$$

are of bounded variation in the infinite interval, then

$$|\Sigma_n(x) - \phi(x)| < Mn^{-k/2}(1 + x^2)^{1/6}e^{-x^2/4},$$

where $\Sigma_n(x)$ is the sum of n terms of (3) and M is a constant independent of n .

If $f(x) = e^{x^2/4}\phi(x)$ then $f(x)$ satisfies the hypotheses of Theorem I, and if we multiply (39) by $e^{-x^2/4}$ we obtain (3). From Theorem I we see that the remainder after n terms of (3) will be

$$O(n^{-k/2})e^{-x^2/4}$$

in the interval $-N \leq x \leq N$ and will be

$$O(n^{-k/2+1/6})e^{-x^2/4}$$

outside this interval. Now when $|x| > N$ it is clear that

$$n^{1/6} < (1 + x^2)^{1/6}$$

whence the truth of Theorem II is apparent.

Since the function $(1 + x^2)^{1/6}e^{-x^2/4}$ is bounded we have

COROLLARY I. *With the hypotheses of Theorem II*

$$\sum_n(x) = \phi(x) + O(n^{-k/2}), \quad -\infty < x < \infty.$$

It will be noted that Theorem I requires $x^{k+1}f(x)$ to be of bounded variation in the infinite interval, which carries with it the implication that for x large

$$f(x) = O(x^{-k-1}).$$

This condition can be somewhat lightened, at the expense however of a corresponding increase in the remainder term for large values of x . We suppose that $f(x)$ satisfies the hypotheses of Theorem I, let

$$f(x) = S_n(x) + R_n(x),$$

multiply this by $(1+x^2)$ and get rid of the x^2 in the terms of the series by two successive applications of the formula

$$xW_p(x) = -(p+1)^{1/2}W_{p+1}(x) - p^{1/2}W_{p-1}(x).$$

Then we have

$$f_1(x) = C'_0 W_0(x) + C'_1 W_1(x) + \cdots + C'_{n-2} W_{n-2}(x) + R_{1,n}(x),$$

in which

$$f_1(x) = (1+x^2)f(x),$$

$$C'_p = [(p-1)p]^{1/2}C_{p-2} + (2p+2)C_p + [(p+1)(p+2)]^{1/2}C_{p+2},$$

and

$$\begin{aligned} R_{1,n}(x) = & \{ [(n-2)(n-1)]^{1/2}C_{n-3} + 2nC_{n-1} \} W_{n-1}(x) \\ & + \{ [(n-1)n]^{1/2}C_{n-2} + (2n+2)C_n \} W_n(x) \\ & + \{ [n(n+1)]^{1/2}C_{n-1} \} W_{n+1}(x) \\ & + \{ [(n+1)(n+2)]^{1/2}C_n \} W_{n+2}(x) \\ & + R_n(x)(1+x^2). \end{aligned}$$

In the course of the proof of Theorem I we have shown that

$$C_n = O(n^{-k/2-3/4})$$

and this fact together with (25), (26), and the inequality

$$n^{1/6} < (1+x^2)^{1/6}, \quad |x| > N,$$

enables us to show that

$$R_{1,n}(x) = O(n^{-k/2})(1+x^2)^{7/6}.$$

Use of the recursion formula shows that the coefficients which we have designated by C'_p are actually the coefficients which we would have obtained

in the formal expansion of the function $f_1(x)$ itself, and therefore $R_{1,n}(x)$ is the remainder after n terms of the series (39) formed for the function $f_1(x)$. Therefore we have the following modification of Theorem I.

THEOREM III. *If $f(x)$ has a continuous k th derivative of bounded variation, and if $x^{k-1}f(x)$, $x^{k-2}f'(x)$, $x^{k-3}f''(x)$, \dots , $x^{-1}f^{(k)}(x)$ are also of bounded variation when x is large, then*

$$S_n(x) = f(x) + O(n^{-k/2})(1+x^2)^{7/6},$$

for all values of x .

It is this theorem which is to be regarded as closest analogue to the theorem of Jackson on Sturm-Liouville series referred to above.

The statement and proof of the corresponding theorem for the case of the series (3) may be left to the reader.

7. Since the results obtained in the preceding paragraph indicate a rate of convergence much slower than in the corresponding situation for Fourier series the question naturally arises whether these conclusions are inevitable or are merely due to inadequate methods of proof. The following example shows that no substantial improvement in the conclusions stated by the theorems need be expected.

Let $f(x) = |x|e^{-x^2/4}$. This function satisfies the hypotheses of Theorem I for $k=1$ except that the derivative has a finite discontinuity at $x=0$. But we have seen that the proof of Theorem I can still be carried through in spite of a finite number of discontinuities of the derivative. We multiply the identity (13) by the function $|s|e^{-s^2/4}$, integrate with respect to s from $-\infty$ to ∞ , assume that n is odd, then set $x=0$, and have

$$S_n(0) = n^{1/2}W_{n-1}(0) \left[\int_{-\infty}^0 - \int_0^{\infty} \right] e^{-s^2/4} W_n(s) ds.$$

From equations (1) and (8)

$$n^{1/2} \left[\int_{-\infty}^0 - \int_0^{\infty} \right] e^{-s^2/4} W_n(s) ds = 2W_{n-1}(0)$$

so that

$$S_n(0) = 2W_{n-1}(0) = 2(2\pi)^{-1/2} [1 \cdot 3 \cdot 5 \cdots (n-2) / (2 \cdot 4 \cdot 6 \cdots (n-1))].$$

The use of Stirling's formula shows that this last expression is equal to

$$(2/\pi)n^{-1/2} + O(n^{-3/2}).$$

Since $f(0) = 0$

$$|S_n(0) - f(0)| > (1/\pi)n^{-1/2}.$$

This example therefore shows that a stronger conclusion than the one stated for Theorem I is not to be expected.

As a second example designed to show that the hypotheses about vanishing at infinity cannot be dispensed with, consider the function $f(x) = 1$, which satisfies all the conditions of Theorem I for any k except that of vanishing at infinity. We note that $C_n = 0$ if n is odd, and C_n is twice the expression given by (17) if n is even. Therefore if we integrate (18) from $-\infty$ to x and compare coefficients with the above values of C_n we find that

$$S_n(x) = (\pi/2)^{1/4} [1 \cdot 3 \cdot 5 \cdots n / (2 \cdot 4 \cdot 6 \cdots (n-1))]^{1/2} \int_{-\infty}^x W_n(x) dx,$$

or

$$S_n(x) = n^{1/4} (1 + \epsilon_n) \int_{-\infty}^x W_n(x) dx,$$

where n is odd. From this it is plain that

$$\lim_{x=\pm\infty} S_n(x) = 0,$$

so that for any given n we may always choose x so large that

$$f(x) - S_n(x) > 1 - \epsilon, \quad \epsilon > 0.$$

Thus no relation of the form

$$|f(x) - S_n(x)| = O(n^{-1/2})$$

can hold uniformly in the infinite interval.

We may remark however that by means of (36) and (20) it may be shown that

$$\begin{aligned} (41) \quad S_n(x) &= n^{1/4} (1 + \epsilon_n) \left[\int_{-\infty}^0 W_n(x) dx + \int_0^x W_n(x) dx \right] \\ &= 1 + O(n^{-1/2}) + n^{1/4} (1 + \epsilon_n) \int_0^x W_n(x) dx \\ &= 1 + O(n^{-1/2}), \quad -N \leq x \leq N, \\ &= 1 + O(1), \quad -\infty < x < \infty. \end{aligned}$$

8. Theorems I, II, and III have stated sufficient conditions for a given degree of convergence. The next two will furnish certain necessary conditions. The first of these two theorems is closely analogous to one given by Jackson* for the case of Fourier series and shows that the existence of deriva-

* Jackson, *Über die Genauigkeit der Annäherung stetiger Funktionen durch rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung*, Dissertation, Göttingen, 1911; Satz XVIII.

tives of certain orders is necessary for a given degree of convergence.

THEOREM IV. *If a series*

$$S(x) = A_0W_0(x) + A_1W_1(x) + A_2W_2(x) + \cdots$$

converges so that for any n its remainder after n terms is

$$R_n(x) + O(n^{-k-\epsilon}), \quad k \text{ an integer, } \epsilon > 0,$$

uniformly for x in any interval, then the sum $S(x)$ has a continuous derivative of order $2k$ in the same interval.

Consider the sum

$$\sum_{m=n}^{n+p} m^q A_m W_m(x) = \sum_{m=n}^{n+p} m^q [R_m(x) - R_{m+1}(x)].$$

By partial summation this may be put in the form

$$\sum_{m=n}^{n+p+1} [(m+1)^q - m^q] R_{m+1}(x) + n^q R_n(x) - (n+p)^q R_{n+p+1}(x).$$

Now

$$(m+1)^q - m^q < q(m+1)^{q-1}.$$

Hence the general term under the sign of summation is less in absolute value than

$$Mq(m+1)^{q-1-k-\epsilon}, \quad M \text{ a constant,}$$

from which we see that the sum itself is

$$O(n^{q-k-\epsilon}).$$

The two terms not under the summation sign are clearly also

$$O(n^{q-k-\epsilon})$$

so that

$$\sum_{m=n}^{n+p} m^q A_m W_m(x) = O(n^{q-k-\epsilon}).$$

From this we conclude that the set of series

$$\sum_{m=0}^{\infty} m^q A_m W_m(x) \quad (q = 1, 2, \cdots, k)$$

all converge uniformly in the given interval.

Now from equation (12)

$$\begin{aligned}\sum_{m=0}^{\infty} A_m W_m''(x) &= - \sum_{m=0}^{\infty} A_m (m + 1/2 - x^2/4) W_m(x) \\ &= - \sum_{m=0}^{\infty} m A_m W_m(x) + (x^2/4 - 1/2) \sum_{m=0}^{\infty} A_m W_m(x).\end{aligned}$$

From the uniform convergence of the two series on the right follows the uniform convergence of the series of second derivatives, which in turn establishes the existence and continuity of the second derivative of $S(x)$. To proceed, we differentiate twice the relation

$$S''(x) = (x^2/4 - \frac{1}{2})S(x) - \sum_{n=0}^{\infty} n A_n W_n(x),$$

substitute for $W_n''(x)$ from (12), and obtain formally

$$\begin{aligned}S^{(4)}(x) &= (x^2/4 - \frac{1}{2})S''(x) + xS'(x) + \frac{1}{2}S(x) \\ &\quad + (\frac{1}{2} - x^2/4) \sum_{n=0}^{\infty} n A_n W_n(x) + \sum_{n=0}^{\infty} n^2 A_n W_n(x).\end{aligned}$$

The uniform convergence of the series involved in this last relation shows that the differentiation was legitimate and establishes the existence and continuity of $S^{(4)}(x)$. In the same manner successive differentiations and substitutions establish the existence of the derivatives of even order up to $S^{(2k)}(x)$. It is only necessary to eliminate by means of preceding equations all of the series which appear with coefficients involving x , since otherwise the double differentiation would introduce terms containing $W_n'(x)$, which cannot be expressed in terms of $W_n(x)$.

COROLLARY. *In a fixed finite interval Theorem IV may be applied to the series (3) without change of wording.*

Since we may take our interval as large as we like, it appears that the corollary establishes the existence of derivatives up to order $2k$ for all values of x provided the hypotheses hold uniformly in the infinite interval.

Our second theorem furnishing necessary conditions deals with the behavior of the function at infinity, and is as follows:

THEOREM V. *If the series*

$$S(x) = A_0 W_0(x) + A_1 W_1(x) + A_2 W_2(x) + A_3 W_3(x) + \cdots$$

converges so that the remainder after n terms is $O(n^{-k/2})$ uniformly in the infinite interval, then

$$S(x) = O(|x|^{-k})$$

when $|x|$ is large.

It may be verified without difficulty that the functions $W_n(x)$, $W'_{n-1}(x)$, $W_{n-2}(x)$ all have the same sign when $x > h$, so that from the relationship

$$n^{1/2}W_n(x) - (n-1)^{1/2}W_{n-2}(x) = 2W'_{n-1}(x),$$

obtained from (11), we derive the inequality

$$|W_{n-2}(x)| < n^{1/2}(n-1)^{-1/2} |W_n(x)|.$$

By successive repetitions this gives us

$$|W_{n-2m}(x)| < [n(n-2) \cdots (n-2m+2)]^{1/2} \\ \cdot [(n-1)(n-3) \cdots (n-2m+1)]^{-1/2} |W_n(x)| < n^{1/2} |W_n(x)|.$$

Likewise

$$|W_{n-2m+1}(x)| < n^{1/2} |W_{n-1}(x)|.$$

Now if we let $X = h + 2k + 6$ and choose $|x| \geq X$ we may apply inequality (29), with the exponent $-k/2 - 3/2$, to the functions $W_{n-1}(x)$ and $W_n(x)$, so that in view of the inequalities just obtained above

$$(42) \quad W_m(x) = O(n^{-k/2-1}), \quad m = 0, 1, 2, \dots, n, \quad |x| \geq X,$$

and from this follows

$$S_n(x) = O(n^{-k/2}).$$

By hypothesis

$$S(x) = S_n(x) + O(n^{-k/2}),$$

so that

$$S(x) = O(n^{-k/2}) = O(X^{-k}).$$

Since this result is true in particular when $|x| = X$, we easily draw the conclusion of the theorem. The fact that X takes on isolated values only does not affect the truth of the theorem for the continuous variable x .

9. Up to the present we have been concerned with the degree of convergence of the expansions (3) or (39) in which the coefficients are determined by the formulas (4) or (40). Now, however, we take up the case in which the function $f(x)$ is to be expressed as a linear combination of W_0, W_1, \dots, W_n , with coefficients which are not necessarily given by (40). The first result is

THEOREM VI. *If $f(x)$ satisfies a Lipschitz condition*

$$|f(x_1) - f(x_2)| < L|x_1 - x_2|$$

and if

$$|xf(x)| < L,$$

then there exists a sum

$$(43) \quad \Sigma_n(x) = A_{0n}W_0(x) + A_{1n}W_1(x) + \cdots + A_{nn}W_n(x)$$

such that

$$|f(x) - \Sigma_n(x)| < BLn^{-1/2},$$

in which B denotes a constant independent of n and of $f(x)$.

We shall prove the theorem only for even values of n , $n = 2m$, though it will be clear that the result holds without this restriction. Our starting point is a theorem due to Jackson* which asserts the existence of a polynomial $P_{1m}(x)$ of degree m at most such that in a closed interval of length J

$$|f(x) - P_{1m}(x)| < B_1 L J m^{-1}.$$

Here as in subsequent formulas B_1, B_2 , etc. denote constants independent of n and of $f(x)$. We shall choose the interval

$$-X \leq x \leq X, \quad X = h + 8,$$

so that

$$J = 2[(8m + 2)^{1/2} + 8] = (32m)^{1/2}[1 + O(m^{-1/2})].$$

We may therefore write

$$(44) \quad P_{1m}(x) = f(x) + \rho_{1m}(x),$$

where

$$|\rho_{1m}(x)| < B_2 L m^{-1/2}.$$

We next turn to the expansion of unity in a series $S_m(x)$ as given by (41). Clearly we have

$$S_m(x) = e^{-x^2/4} P_{2m}(x),$$

where $P_{2m}(x)$ is a polynomial of degree m at most, so that (41) can be put in the form

$$(45) \quad e^{-x^2/4} P_{2m}(x) = 1 + \rho_{2m}(x)$$

with the conditions

$$|\rho_{2m}(x)| < B_3 m^{-1/2}, \quad -N \leq x \leq N,$$

and

$$|\rho_{2m}(x)| < B_4, \quad -\infty < x < \infty.$$

Now we multiply (44) and (45), obtaining

$$e^{-x^2/4} P_n(x) = f(x) + R_n(x),$$

in which $P_n(x)$ is a polynomial of degree n at most. From the inequalities satisfied by $\rho_{1n}(x)$ and $\rho_{2n}(x)$ together with the fact that $|xf(x)| < L$ we readily deduce that

* These Transactions, vol. 14 (1913), p. 351, Theorem V.

$$(46) \quad |R_n(x)| < BLn^{-1/2}.$$

Since $e^{-x^2/4}P_n(x)$ is a linear combination of W_0, W_1, \dots, W_n , we may set

$$e^{-x^2/4}P_n(x) = \Sigma_n(x)$$

and have

$$(47) \quad \Sigma_n(x) = f(x) + R_n(x)$$

in the interval $-X \leq x \leq X$, so that the desired theorem is established in this interval.

Now we consider the magnitude of the remainder term for x outside of this interval. First of all we observe by means of (42) for the case $k=0$, that

$$\int_{-X}^X W_i(x)W_j(x)dx = \begin{cases} 1 + O(n^{-2}), & i = j \leq n, \\ O(n^{-2}), & i < j \leq n. \end{cases}$$

If A_{jn} denotes the largest coefficient in (43) we multiply (47) by $W_j(x)$, integrate from $-X$ to X , and have

$$A_{jn}(1 + O(n^{-1})) = \int_{-X}^X f(x)W_j(x)dx + \int_{-X}^X W_j(x)R_n(x)dx,$$

from which it appears that A_{jn} is bounded with respect to n . For from (26) and (29) it is seen that $|x^{1/6}W_j(x)|$ is bounded, and the same is true of $|x f(x)|$, so that the first integral is bounded. That the second integral is bounded is obvious from (46).

Finally we use (42) again for the case $k=1$ getting

$$|W_p(x)| < B_5 n^{-3/2} \quad (p = 0, 1, \dots, n),$$

whence

$$|\Sigma_n(x)| < B_6 Ln^{-1/2},$$

and since $|f(x)| < Ln^{-1/2}$ when $|x| > X$, we see that, for a suitable B , (46) holds for all values of x , and the theorem is established for the infinite interval.

If x is replaced by $2^{1/2}x$ a sum of the type (43) is converted into a sum of the type

$$(48) \quad \sigma_n(x) = b_0\phi_0(x) + b_1\phi_1(x) + \dots + b_n\phi_n(x),$$

whence we have

THEOREM VII. *Theorem VI also holds for the sum (48).*

On the other hand the introduction of the factor $e^{-x^2/4}$ enables us to deduce from Theorem VI the following

THEOREM VIII. If $\phi(x)$ satisfies a condition of the form

$$|e^{x_1^2/4}\phi(x_1) - e^{x_2^2/4}\phi(x_2)| < L|x_1 - x_2|$$

and if

$$|xe^{x^2/4}\phi(x)| < L$$

there exists a sum of the type (48) such that

$$|\phi(x) - \sigma_n(x)| < BLn^{-1/2}e^{-x^2/4}.$$

10. We are now in a position to establish a theorem very similar to one proved by Jackson* for the indefinite integral of a trigonometric sum.

THEOREM IX. Let $\phi(x)$ be an integrable function for which

$$(49) \quad \int_{-\infty}^{\infty} \phi(x)dx = 0,$$

and suppose that there exists a sum $\sigma_n(x)$ of the form given by (48) such that

$$(50) \quad \phi(x) = \sigma_n(x) + \rho_n(x),$$

where

$$|\rho_n(x)| < \epsilon e^{-x^2/4}$$

for all values of x in the infinite interval. Then

$$(51) \quad \int_{-\infty}^x \phi(x)dx = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + r_n(x)$$

in which

$$|r_n(x)| < 5B\epsilon n^{-1/2}e^{-x^2/4}.$$

The proof is so similar to that of Jackson that it might be omitted, were it not that the behavior of the remainder terms when x is large requires some slight additional investigation. We shall sketch the proof briefly. If we integrate (50) from $-\infty$ to ∞ and use (49) we find that the term $b_0\phi_0$ can be included with $\rho_n(x)$ to form a new remainder $\rho'_n(x)$ for which

$$|\rho'_n(x)| < (2^{1/2} + 1)\epsilon e^{-x^2/4}.$$

We then integrate (50) from $-\infty$ to x , obtaining

$$(52) \quad \int_{-\infty}^x \phi(x)dx = b_1\phi_0(x) + b_2\phi_1(x) + \cdots + b_n\phi_{n-1}(x) + R_n(x)$$

in which

$$R_n(x) = \int_{-\infty}^x \rho'_n(x)dx.$$

* These Transactions, vol. 13 (1912), pp. 491-515.

Clearly $R_n(x)$ vanishes at $-\infty$ and in view of (49) it also vanishes at ∞ . Therefore

$$R_n(x) = - \int_x^\infty \rho_n'(x) dx.$$

If x is negative we use the first form of $R_n(x)$, but if x is positive we use the second form. Suppose that x is positive. Then

$$\begin{aligned} |R_n(x)| &< (2^{1/2} + 1)\epsilon \int_x^\infty e^{-x^2/4} dx \\ &< (2^{1/2} + 1)\epsilon x^{-1} \int_x^\infty e^{-x^2/4} x dx \\ &= 2(2^{1/2} + 1)\epsilon x^{-1} e^{-x^2/4} < 5\epsilon x^{-1} e^{-x^2/4}. \end{aligned}$$

The same is true when x is negative. Thus we have shown that $R_n(x)$ satisfies the second part of the hypothesis of Theorem VIII with $L = 5\epsilon$. That $R_n(x)$ satisfies the first part of the hypothesis with $L = 5\epsilon$ follows from the fact, readily shown from the foregoing inequalities, that the derivative of $e^{x^2/4} R_n(x)$ does not exceed 5ϵ in absolute value.

Therefore from Theorem VIII we see that there exists a sum $s_n(x)$ involving the first n ϕ 's, such that

$$|R_n(x) - s_n(x)| < 5B\epsilon n^{-1/2} e^{-x^2/4}.$$

We use this result in connection with (52), combine the two sums and obtain (51).

From the two theorems VIII and IX we derive

THEOREM X. *If $\phi(x)$ has a $(k-1)$ th derivative which satisfies the conditions*

$$|e^{x_1^2/4} \phi^{(k-1)}(x_1) - e^{x_2^2/4} \phi^{(k-1)}(x_2)| < L |x_1 - x_2|,$$

$$(53) \quad |xe^{x^2/4} \phi^{(k-1)}(x)| < L,$$

and if

$$(54) \quad \lim_{x=\pm\infty} \phi(x) = 0,$$

then there exists a sum $\sigma_n(x)$ of the form (48) such that

$$|\phi(x) - \sigma_n(x)| < (5B)^k L n^{-k/2} e^{-x^2/4}.$$

To establish this we first apply Theorem VIII to the function $\phi^{(k-1)}(x)$, then use Theorem IX to obtain an approximating sum for $\phi^{(k-2)}(x)$, again use IX to obtain an approximating sum for $\phi^{(k-3)}(x)$, and continue thus till

the desired result is reached for $\phi(x)$. In making these applications of Theorem IX we need to know, in addition to the stated hypotheses, that

$$\int_{-\infty}^{\infty} \phi^{(m)}(x) dx = 0 \quad (m = 1, 2, \dots, k-1),$$

or what amounts to the same thing, that

$$\lim_{x=\pm\infty} \phi^{(m)}(x) = 0 \quad (m = 0, 1, 2, \dots, k-2).$$

Now if x is large and positive

$$\begin{aligned} \phi^{(k-2)} &= C - \int_x^{\infty} \phi^{(k-1)}(x) dx \\ &= C + O(x^{-2}e^{-x^2/2}), \\ \phi^{(k-3)} &= Cx + C' + O(x^{-3}e^{-x^2/2}). \end{aligned}$$

Continuing thus we find that $\phi(x)$ may be expressed as a polynomial with an added term which is $O(x^{-k}e^{-x^2/2})$. But the condition (54) shows that the polynomial must vanish identically, and therefore C, C' , etc. all vanish, which completes the proof.

If in Theorem X we make the change of function

$$\phi(x) = e^{-x^2/4}f(x),$$

we are enabled to state a theorem regarding approximation by sums of the type (41):

THEOREM XI. *If $f(x)$ has a $(k-1)$ th derivative which satisfies a Lipschitz condition*

$$|f^{(k-1)}(x_1) - f^{(k-1)}(x_2)| < L|x_1 - x_2|,$$

and if

$$|x^{k-m}f^{(m)}(x)| < L \quad (m = 0, 1, 2, \dots, k-1),$$

then there exists a sum $\Sigma_n(x)$ such that

$$\Sigma_n(x) = f(x) + O(n^{-k/2})$$

uniformly in the infinite interval.

The proof of this theorem may be left to the reader.

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